iid with unknown distributions, contined...

Note Title 1/31/2013

- Budgeted Allocation problem. Say $\frac{B_i}{b_i^{max}} \geq k \; \forall i$

$$\text{OPT} = \max \sum_{i,j} b_{ij} x_{ij} \text{ s.t.}$$

m.p. p_j

$$\forall i \quad \sum_j b_{ij} x_{ij} \leq B_i$$

$$\forall j \quad \sum_i x_{ij} \leq 1. \quad \left[\text{Assume } p_j = \frac{1}{m}\right]$$

$$x_{ij} \geq 0.$$

Assume that for the optimal soln. to), all Budgets B_i's are exhausted.

Pure Random: Given j, match it to i with prob. x_{ij}. (opt. soln.)

X_i = spend of i = $m \cdot X_i^t$ = $m \cdot$ spend of i in step t

$$= m \cdot \sum_j \frac{1}{m} b_{ij} x_{ij} = B_i$$

$E[\min\{X_i, B_i\}]$ is minimized when $\forall j \;\; b_{ij} \in \{0, b_i^{max}\}$ & $m \to \infty$.

In this case, it is $\approx B_i - \sqrt{\frac{B_i}{2\pi}}$

Thus $E[PR] = \sum_i B_i - \sqrt{B_i/2\pi}$

Algo w/o distribution knowledge: Inductive defn.

Given $A_1, \ldots A_{t-1}$? $P_{t+1}, \ldots \ldots P_m = H^t$

Also assume $b_{ij} \in \{0, b_i^{max}\}$

With this assumption, given
- remaining budget for i, $\hat{B}_i$
- remaining steps $(m-t)$.
- In each step i gets b_i^{max} w.p. $\frac{B_i}{b_i^{max} \cdot m}$.

Can estimate expected revenue from running PR for remaining steps.

$\therefore$ Match j to $\arg\max_i \left\{ b_{ij} + E[P_{t+1}, \dots P_m \mid A_t = i] \right\}$

As before, $E[H^t] \geq E[H^{t-1}]$

$\therefore E[ALG] \geq E[PR]$

$\square$

Open Question:- Can we eliminate the assumption that Budgets are exhausted?

For example, making $B_i \to \infty$ for some i should only help.

Resource Allocation Problem:- (Recall)

i = resource, capacity C_i

j = request, feasible options F_j.

$\forall k \in F_j$, $\forall i$, consume $a(i,j,k)$.

profit $= w(j,k)$

LP: $\max \sum_{j,k} w(j,k)\, x(j,k)$ s.t.

$\forall i \quad \sum_{j,k} a(i,j,k)\, x(j,k) \leq C_i,$

$\forall j \quad \sum_k x(j,k) \leq 1, \quad x(j,k) \geq 0.$

iid model: distribution on j's.
As before assume for simplicity that $p(j) \geq \frac{1}{M}$ $\forall j$.

Goal: Design algo s.t. w.p $1-\delta$, $ALG \geq (1-\varepsilon)\,\overline{OPT}$.
(High probability vs. in expectation)

Suppose $\frac{C_i}{a(i,j,k)} \geq \cancel{k}$. As before assume we know m.
Also $OPT / w(j,k) \geq k$.

Now, also assume we know the value OPT

First, consider ~~Pure~~ Random:

- Given j, use option k w.p. $\frac{x_{j,k}}{1+\varepsilon}$ (opt. soln.)

Let X_i = total capacity of i consumed by pure random.

W = profit of Pure random

n = # of resources.

Claim:- $\forall \varepsilon, \delta > 0$, If $\cancel{k} \geq \frac{c \cdot \log(n/\delta)}{\varepsilon^2}$ for some universal constant c,

Then w.p $1-\delta$, $\forall i$ $X_i \leq C_i$

& $W \geq (1-2\varepsilon)\,\overline{OPT}$.

For simplicity assume $a(i,j,k) \in [0,1]$, &
$C_i \geq k$. lly $w(j,k) \in [0,1]$ & $\overline{OPT} \geq k$.

Chernoff Bounds

$$X_i = \sum_{t=1}^{m} X_{i,t}.$$

$$E[X_{i,t}] = \sum_j \frac{1}{m}\cdot\frac{x_{ij}}{1+\varepsilon}\, a(i,j,k) \le \frac{C_i}{m(1+\varepsilon)}$$

$$E[X_i] \le C_i/(1+\varepsilon)$$

$$Pr[X_i \ge C_i] = Pr\left[(1+\varepsilon)^{X_i} \ge (1+\varepsilon)^{C_i}\right]$$

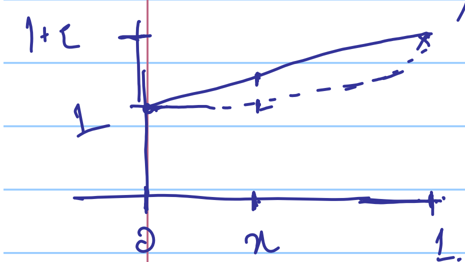
(By Markov's Ineq. $Pr[X \ge a] \le E[X]/a$)

$$\le E\left[(1+\varepsilon)^{X_i}\right] / (1+\varepsilon)^{C_i}$$

$$= E\left[\prod_{t=1}^{m} (1+\varepsilon)^{X_{i,t}}\right] / (1+\varepsilon)^{C_i}$$

(Independence across t's)

$$= \prod_{t=1}^{m} E\left[(1+\varepsilon)^{X_{i,t}}\right] / (1+\varepsilon)^{C_i}$$



$$\le \prod_{t=1}^{m} E\left[1+\varepsilon X_{i,t}\right] / (1+\varepsilon)^{C_i}$$

$$\le \prod_{t=1}^{m} \left(1+\frac{\varepsilon C_i}{(1+\varepsilon)m}\right) / (1+\varepsilon)^{C_i}$$

∵ $1+x \le e^{x}$

$$\le \prod_{t=1}^{m} e^{\varepsilon C_i/(1+\varepsilon)m} / (1+\varepsilon)^{C_i}$$

$$= e^{\varepsilon C_i/(1+\varepsilon)} / (1+\varepsilon)^{C_i}$$

∵ $\frac{e^{x}}{(1+x)^{(1+x)}} \le e^{-x^2/2}$

$$= \left[\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right]^{\frac{C_i}{1+\varepsilon}}$$

$$\le e^{-\frac{\varepsilon^2}{4}\cdot C_i}$$

∵ $C_i \ge \frac{c\cdot \log\frac{n}{\delta}}{\varepsilon^2}$

$$\le e^{-\log(2n/\delta)} \le \delta/2n$$

III^{ly} $\Pr\left[W < \frac{\overline{OPT}}{(1+\varepsilon)}(1-\varepsilon) \right]$ $W = \sum_t W^t$ = Profit of PR

$$\le \Pr\left[(1-\varepsilon)^{W} > (1-\varepsilon)^{\frac{\overline{OPT}(1-\varepsilon)}{(1+\varepsilon)}} \right]$$

$$\le \mathbb{E}\left[(1-\varepsilon)^{W} / (1-\varepsilon)^{\overline{OPT}(1-\varepsilon)/(1+\varepsilon)} \right]$$

$$= \prod_{t=1}^{m} \mathbb{E}\,(1-\varepsilon)^{W_t} / (1-\varepsilon)^{\cdots}$$

$$\le \prod_{t=1}^{m} \mathbb{E}(1-\varepsilon W_t) / \cdots$$

$$\le \prod_{t=1}^{m} \left(1 - \varepsilon \frac{\overline{OPT}}{(1+\varepsilon)m}\right) / \cdots$$

$$\le \prod_{t=1}^{m} e^{-\varepsilon \overline{OPT}/(1+\varepsilon)m} / \cdots$$

$$= e^{-\varepsilon \overline{OPT}/(1+\varepsilon)} / (1-\varepsilon)^{\overline{OPT}(1-\varepsilon)/(1+\varepsilon)}$$

$\vdots$

$$\le e^{-\frac{\varepsilon^2 \overline{OPT}}{2(1+\varepsilon)}}$$

$$\le e^{-\log \frac{2n}{\delta}} \le \delta/2n.$$

$\Pr[X_i \geq c_i] = \Pr\left[\frac{k X_i}{c_i} \geq k\right] \leq \frac{E\left[(1+\varepsilon)^{\frac{kX_i}{c_i}}\right]}{(1+\varepsilon)^k} \leq \frac{\delta}{n}$

If $\sum_i \frac{E\left[(1+\varepsilon)^{kX_i/c_i}\right]}{(1+\varepsilon)^k} \leq \delta$, then

$E\left[\max_i (1+\varepsilon)^{kX_i/c_i}\right] / (1+\varepsilon)^k \leq \delta \Rightarrow$

$\Pr\left[\max_i (1+\varepsilon)^{kX_i/c_i} \geq (1+\varepsilon)^k\right] \leq \delta \Rightarrow$

$\Pr[\exists i : X_i \geq c_i] \leq \delta.$ $\equiv$ Union bound

... cont'd ....

PR can be thought of as minimizing

$$\bar{\Phi} = \sum_i \frac{(1+\varepsilon)^{KX_i/C_i}}{(1+\varepsilon)^K} + \frac{(1-\varepsilon)^{KW/OPT}}{(1-\varepsilon)^K}$$ & $\leq \delta$ in expectation.

Idea:- Algo also minimizes the same.
Using Hybrid argument, show that

$$E[\bar{\Phi}^A] \leq E[\Phi^P] \leq \delta.$$

(Actually an "upper bound" on these expectations)

Consider $H^T = A_1 A_2 \ldots A_T P_{T+1} \ldots P_m.$

$$E[\Phi^{H^T} \mid A_1, A_2, \ldots, A_{T-1}]$$

$$(1+\varepsilon)^{X_i^{H^T}} = \prod_{t=1}^{T-1} (1+\varepsilon)^{X_{i,t}^A} \; (1+\varepsilon)^{X_{i,T}^A} \cdot \prod_{t=T+1}^{m} (1+\varepsilon)^{X_{i,t}^P}$$

$X_{i,t}^P \;\forall\, t = T+1 \ldots m$ are independent of each other, and of $X_{i,T}^A$.

$$\prod_{t=T+1}^{m} E(1+\varepsilon)^{X_{i,t}^P} \leq \ldots \ldots$$

$$\leq \prod_{t=T+1}^{m} e^{\varepsilon C_i/(1+\varepsilon)\cdot m} = e^{\frac{\varepsilon C_i (m-T)}{(1+\varepsilon) m}}$$

$$E\left[\frac{(1+\varepsilon)^{kx_i^{ALG}/c_i}}{(1+\varepsilon)^k} \,\middle|\, A_1, \ldots, A_{T-1} \right] \leq$$

$$\underbrace{\prod_{t=1}^{T-1} (1+\varepsilon)^{\frac{kx^A_{i,t}}{c_i}} \cdot \frac{e^{\frac{\varepsilon k (m-T)}{(1+\varepsilon) m}}}{(1+\varepsilon)^k}}_{\Phi_{i,T-1}} \cdot E\left[(1+\varepsilon)^{\frac{kx^A_{i,T}}{c_i}} \right]$$

$$\leq \Phi_{i,T-1} \, E\left[1 + \frac{\varepsilon k x^A_{i,T}}{c_i} \right]$$

$$= \Phi_{i,T-1} + E\left[\varepsilon k \Phi_{i,T-1} \cdot \frac{x^A_{i,T}}{c_i} \right]$$

$$E[\phi^{ALG}] \leq \sum_i \text{———} \;\, '' \;\, \text{———}$$

$$+ \Phi_{w,T-1} - E\left[\varepsilon k \phi_{w,T-1} \frac{w^A_T}{OPT} \right]$$

Want to minimize

$$\sum_i \Phi_{i,T-1} \frac{x^\Delta_{i,T}}{c_i} - \phi_{w,T-1} \frac{w^\Delta_T}{OPT}$$

$\therefore$ given j choose option

$$\arg\min_k \left\{ \sum_i \Phi_{i,T-1} \frac{a_{i,j,k}}{c_i} - \Phi_{w,T-1} \frac{w_{jk}}{OPT} \right\}$$

$$= \arg\max_k \left\{ w_{jk} - \sum_i \alpha_i a_{ijk} \right\} \quad \text{where}$$

$$\alpha_i = \Phi_{i,T-1}/c_i \times OPT/\Phi_{w,T-1}$$

dual

The min. choice is better than "Pure random choice".

$\therefore E\left[\phi^{A^T} \mid A_1 \ldots A_{T-1}\right] \leq \sum_i \phi_i^{T-1} \cdot e^{\varepsilon k/(1+\varepsilon)m} + \phi_w^{T-1} \cdot e \ldots$

As good as the upper bound on $E\left[\phi^{H^{T-1}} \mid A_1, \ldots A_{T-1}\right]$ □.

$$\phi_{i,0} = \frac{e^{\frac{\varepsilon k(m-1)}{(1+\varepsilon)m}}}{(1+\varepsilon)^k}$$

$$\phi_{i,T} = \phi_{i,T-1} \cdot \frac{(1+\varepsilon)^{kx_{i,T}/c_i}}{e^{\varepsilon k/(1+\varepsilon)m}}$$

$$\simeq \phi_{i,T-1} \cdot e^{\varepsilon k\left[\frac{x_{i,T}}{c_i} - \frac{1}{(1+\varepsilon)m}\right]}$$

Actual Ideal

Simple dual-update rule.